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## Retail Competition with Switching Consumers in Electricity Markets

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### Abstract

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The ongoing transformations of power systems worldwide pose important challenges, both economic and technical, for their appropriate planning and operation. A key approach to improve the efficiency of these systems is through demand-side management, i.e., to promote the active involvement of consumers in the system. In particular, the current trend is to conceive systems where electricity consumers can vary their load according to real-time price incentives, offered by retailing companies. Under this setting, retail competition plays an important role as inadequate prices or services may entail consumers switching to a rival retailer. In this work we consider a game theoretical model where asymmetric retailers compete in prices to increase their profits by accounting for the utility function of consumers. Consumer preferences for retailers are uncertain and distributed within a Hotelling line. We analytically characterize the equilibrium of a retailer duopoly, establishing its existence and uniqueness conditions. Furthermore, sensitivities of the equilibrium prices with respect to relevant model parameters are also provided. The duopoly model is extended to a multiple retailer case for which we perform an empirical analysis via numerical simulations. Results indicate that, depending on the retailer costs, loyalty rewards and initial market shares, the resulting equilibrium can range from complete competition to one in which a retailer has a leading or even a dominant position in the market, decreasing the consumers' utility significantly. Moreover, the retailer network configuration also plays an important role in the competitiveness of the system.

**Keywords:** *Elastic consumers, electricity market, hotelling line, market equilibrium, retail competition, switching consumers.*

# Retail Competition with Switching Consumers in Electricity Markets

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## 1. Introduction

Over the last decades, electricity systems worldwide have experienced a set of transformations that have notably modified their structure, operation and efficiency. In general, vertically integrated systems, based on fossil-fuels technologies and with inelastic consumers, have evolved into competitive markets with high penetration of renewable production and price-responsive consumers. These transformations pose both economical and technical challenges to the market and system operators.

From an economical perspective, the recent regularization of the electricity systems worldwide has introduced market mechanisms to incorporate horizontal competition at the generation and

distribution levels. In general, generating companies compete between them, at different trading floors, to supply their energy at the highest prices possible to retailing companies or to large consumers that participate in the market. Similarly, retailers compete with each other to buy their energy at the same trading floors in order to supply it to the final consumers.

From a technical perspective, new generation technologies and consumption patterns are being integrated into the electricity industry. Regarding the supply side of the system, it should be noted the important growth of renewable generation, which is becoming more and more profitable as its investment costs have been decreasing steadily over these past years. While the main advantages of these new energy sources are their low generation and CO<sub>2</sub> emission costs, most of them present a stochastic nature so that their available production capacities are difficult to predict on a short-term basis. The current practice in electricity systems is to use conventional fast-response generation units, for instance, combined-cycle units, to compensate unexpected falls in renewable production and to balance the system (total generation must match total demand load). However, this approach presents several disadvantages related to the early wear of these units and the increase in the final market prices.

One important approach to deal with these challenges is by promoting the active involvement of consumers in the system. In particular, demand response, i.e., the altering of the consumer load patterns as a reaction to specific incentives, is considered one of the main components of this new paradigm (Kirschen 2003). An appropriate demand-side management can smooth load peaks, mitigate short-term imbalances and improve the efficiency of the system. Among several options, the most extended form of demand-side management is to create price incentives to the consumers to switch their load. For instance, high prices can be assigned to peak-hours so that the consumption is moved to different hours. In fact, the European Union and the USA distribution system operators are actually investing a high amount of resources to promote pricing programs such as real-time pricing (RTP), critical-price pricing (CPP) and time-of-use tariffs (Mallet et al. 2014, U.S. Department Energy 2006, U.S. Gov. Policy Act 2005).

Thus, it is expected that in the coming years an important number of consumers would move from systems with fixed consumption tariffs to others in which the corresponding retailing company would offer prices that are updated in real time. Under this framework, if several retailers have access to the same area of the electricity system, they will have to compete to increase their consumer market share. In that case, the retail competition in the electricity market becomes similar to that in other industries, such as communications, banking, private health care, insurance, etc. As electricity is a near-homogeneous good, in addition to the offered price, retailers can also provide complementary services to the consumers to differentiate their products, for instance: reliability of service, information technologies, loyalty programs, etc.

Under this new setting, the role consumers play in the electricity market is increased. They can adapt their consumption to changes in supply and demand and, moreover, they can switch to retail companies with better services or conditions. To allow the growth in their customer base and to retain existing customers, retail companies can offer incentives to reward the loyalty of consumers, or implement switching costs if they decide to switch retailers (Farrel and Klemperer 2007).

In this work, we propose to model this setting as a game where retailers compete in prices to increase their revenues in the long-term. Game-theoretical models have shown to be very suitable to represent these types of competition, specially considering that, in most electricity markets, a small number of retailers rule the market.

Specifically, we analyze the interaction and the possible equilibria between retailers that compete to supply their energy to the final consumers in an electricity market. The aim of each retailer is to maximize its profit: revenues from selling its energy to the consumers minus the costs of purchasing it, by selecting an adequate retail price to offer to the consumers. In particular, we assume that retailers have different purchasing costs as they follow their own strategy to buy this energy, for instance, by participating in the electricity day-ahead market or through bilateral contracts with the generating firms. Moreover, we consider that retailers are able to anticipate, in a Stakelberg fashion, the reaction of the final consumers which seek to maximize their utility functions taking into account the possible loyalty premiums that retailers may offer. Uncertainty is incorporated to the problem by assuming that the consumer preferences for each retailer are uniformly distributed on a Hotelling line (Hotelling 1929).

In summary, the main contributions of this work are sixfold:

1. To propose a game-theoretical model to represent the interactions between several retailers and the final consumers in an electricity market. Retailers aim to maximize their profit while consumers aim to maximize their utility functions.
2. To explicitly account for the loyalty incentives, or switching costs, that consumers may have if they remain with their original retailers.
3. To provide an analytical characterization of the market equilibrium for the case of a retailer duopoly.
4. To establish the existence and uniqueness conditions for a duopoly equilibrium that are valid for a wide class of consumer utility functions.
5. To provide a sensitivity analysis of the equilibrium prices with respect to relevant problem parameters.
6. To generalize this setting for the case of more than two retailers and perform an empirical analysis via numerical simulations.

The article is structured as follows. In Section 2 we provide a literature review of retail competition in electricity markets and an economic analysis of loyalty premiums. Section 3 is dedicated to study and characterize the equilibrium solutions for the duopoly case. Section 4 generalizes this setting to the multiple retailers case. The empirical analysis is carried out in Section 5. Finally, this work's conclusions are drawn in Section 6.

## 2. Literature Review

Many engineering and operations research papers have analyzed the pricing decision process of a retailer in an electricity market. For instance, a real-time pricing framework to ease peak load hours in a smart grid is introduced in Qian et al. (2013), where a single retailer selects the best pricing strategy anticipating the reaction of consumers. Similarly, Bu et al. (2013) proposes a game theoretical model formulated as a four-stage problem to assist retailers in their pricing strategies in a smart grid. Based in time-of-use tariffs, García-Bertrand (2013) presents a risk-based pricing tool for retailers to decrease the consumers' load in those periods where pool prices are high. With a similar aim, a risk-averse two-stage bilevel model is proposed in Wei et al. (2015) to be used by a retailer to derive its optimal dispatch and retail prices. Specifically, a Stakelberg game is considered to model demand response.

Most liberalized electricity markets include more than one retailing company that compete to supply the final consumers. Thus, agent-based and game-theoretical models are generally used to represent these interactions. This is the case of Müller et al. (2007), which presents an agent-based model of the German retail market to study the interdependencies between retailers' pricing strategies and consumers' participation in the market. Joskow and Tyrole (2006) studies the effects of combining retail competition and different consumer profiles. It is shown that, apart from the absence of real-time meters, transaction costs and joint interruptibility can also hamper consumers to react to real-time prices. A game-theoretical model to represent the equilibrium between multiple retailers is introduced in Oliveira et al. (2013). Retailers buy their energy from generators at different trading floors and sell it to the final consumers. However, the retail price is considered unique and consumers are aggregated into a single demand curve.

Although these works notably contribute to address many of the challenges that are faced by retailing firms, issues such as consumer loyalty incentives or switching costs, and their economical implications, are in general not considered in the context of electricity markets. As explained in the previous section, this is mainly because, until very recently, consumers have only exhibited a passive role in electricity systems.

Nevertheless, there is an extensive marketing and economic literature that has addressed the market implications of loyalty incentives or switching costs in other industries. See Farrel and

Klemperer (2007) for a detailed review on this topic. In general, one of the most important questions that arises with these incentives is whether they lead to higher or lower market prices. Two main approaches have been used to answer this question: first, the consideration of analytical models, which aim to reproduce either static or dynamic competition, and second, the use of empirical analysis, based on simulations or in real-world data.

Regarding analytical models, the most widespread approach to represent product differentiation is via a Hotelling line (Hotelling 1929), where consumer preferences are distributed within a line that connects two selling firms. Even if homogeneous products are considered, which is the case for electricity, product differentiation arises from asymmetric transportation costs (in this contest, the concept “transportation costs” is used to quantify the consumer’s difference in preferences for each retailing firm, rather than physical or monetary costs). This model has become an standard model in oligopolistic competition.

Although the original Hotelling model is restricted to the duopoly case, several extensions have been proposed to represent oligopolistic markets. For instance, Chen and Riordan (2007) introduced the spokes model where several oligopolistic firms interact simultaneously in a market with multiple products. In particular, each selling firm competes directly with all other firms, even considering that consumers are only interested in a maximum number of product varieties. An alternative extension of the Hotelling model is presented in Somaini and Einav (2013), where more than two firms can also be considered. The model assumes that each consumer is limited to buy from two asymmetric firms. Due to its similarities with the current electricity retail market, some of these models’ features are used in our numerical analysis to extend the duopoly case.

A dynamic duopoly model is presented in Doganoglu (2010) to study competition between firms. Sufficient conditions are provided for the existence of a Markov equilibrium. Results show that the level of switching costs has an important impact on competition. In particular, when switching costs are low, the resulting prices in the steady state can be lower than without switching costs. A similar analysis is conducted in Rhodes (2014) where the conditions under which switching costs may raise or decrease market prices are established for both the short and long-run. Specifically, it is shown that switching costs are more likely to increase prices in the short-run. An extension of these models is proposed in Chen et al. (2014) where firms in a duopoly are allowed to charge different prices to their own consumers (customized pricing). Results indicate that increasing switching costs (or decreasing loyalty rewards) may decrease firms’ profits.

Many empirical analyses have been carried out to show the impact of switching costs in different industries (see Farrel and Klemperer (2007) for a complete revision). Related to the electricity industry, Waterson (2003) studies the impact of consumers behavior in the industry performance. Several case studies from different industries are analyzed, including the electricity market in the

UK, to show that consumer switching costs have a relevant impact on market efficiency. In the same vein, an empirical analysis to study the impact of switching costs is presented in Wilson and Price (2010). The analysis of some databases from the UK electricity market shows that consumers' ability to select the best retailer is limited. This suggests that, apart from purely economical reasons, there are other factors that may condition consumer switching costs.

Compared to the above works, the main contributions of the proposed model are directly related to considering electricity as our trading good. In particular, we use a general function for the consumer utility which depends on the purchased quantity of electricity, represented as a continuous variable. Note that most of the references assume a fixed quantity (or number of units) of the good to be purchased and the only decision variable represents the choice of an appropriate retailing firm. Additionally, we assume that the retailers are asymmetric as they follow their own strategy when acquiring their electricity at different trading floors (with different prices). Under this general setting we provide analytical results that characterize the equilibrium of the duopoly and extend this framework numerically to a multiple-retailer case study.

### 3. Electricity market with two retailers

Consider an electricity market with two retailers,  $R_i$  and  $R_j$ , that sell their energy at prices,  $p_i$  and  $p_j$ , respectively, to a set of consumers,  $C$ . To model the consumer preferences, we assume that the two retailers are located at the ends of a Hotelling line (Hotelling 1929) of length 1, that is,  $R_i$  is located at the beginning of the line and  $R_j$  at the end. Each consumer is placed at a given position within the Hotelling line  $\theta_{ij} \in [0, 1]$ , so that the closer they are to  $R_i$ , i.e., the smaller  $\theta_{ij}$  is, the higher her preference to buy from  $R_i$ . Moreover, consumers are distributed along this line by following a probability distribution,  $F(\theta_{ij})$ . Additionally, let  $\pi_{ij}$  ( $\pi_{ji}$ ) denote the proportion of consumers with retailer  $R_i$  ( $R_j$ ) at the initial state, where  $\pi_{ij} + \pi_{ji} = 1$ .

For the sake of completeness, we will assume that the consumer utilities include a function  $u$  that is a concave increasing function of the purchased energy  $q_i \geq 0$ , or  $q_j \geq 0$ , if the energy is bought from  $R_i$  or  $R_j$  respectively, with continuous second derivatives. We will also assume that it has an invertible first derivative  $u'(q) \geq 0$ .

The utility maximization problem for a consumer that had bought from  $R_i$  at the initial state, and decides to buy again from retailer  $R_i$ , is given by

$$\max_{q_i} u(q_i) - p_i q_i - k\theta_{ij} + \gamma_i, \quad (1)$$

where the second term represents the payments to  $R_i$ . Parameter  $k > 0$  denotes the transportation cost per unit of length and is used to quantify the preference levels of consumers with respect to each retailer. In particular, the smaller the value of  $\theta_{ij}$  the lower the transportation costs  $k\theta_{ij}$ , yielding an

increased preference for retailer  $R_i$  for this consumer. This preference asymmetry may be caused by the different services offered by the retailers such as: quality and reliability of supply, information technologies, technical support, flexible payments programs, etc. Note that the transportation cost does not depend on the consumer's actual retailing company. Another parameter,  $\gamma_i \geq 0$ , is introduced to represent a *loyalty reward or premium*, i.e., the obtained incentive by purchasing again from  $R_i$ . Note that  $-\gamma_i$  can also be viewed as a switching cost incurred if a consumer decides to switch to  $R_j$ .

In particular, if this same consumer decides to switch to  $R_j$ , the problem she has to solve is

$$\max_{q_j} u(q_j) - p_j q_j - k\theta_{ji}, \quad (2)$$

where  $\theta_{ji} = 1 - \theta_{ij}$  is the Hotelling line distance between the consumer and  $R_j$ . The switching decision is modeled by letting the term  $\gamma_i$  vanish from the utility function.

Similar problems can be obtained if the consumer initially bought from  $R_j$  by interchanging  $i$  and  $j$ .

Regardless of the initial state, the optimal value of  $q_k$ , where  $k = \{i, j\}$ , is obtained from  $u'(q_k) = p_k$  as  $q_k^* = v(p_k) \equiv (u')^{-1}(p_k)$ , where  $v : [\min(c_i, c_j); \alpha] \rightarrow \mathbb{R}^+$ , and we use the notation  $\alpha = u'(0)$ . Under our assumptions,  $v$  is a nonnegative decreasing function of  $p_k$  on its domain; also  $v(\alpha) = 0$ .

Each retailer  $R_i$  or  $R_j$  purchases their energy from the spot or futures market, at a cost  $c_i$  or  $c_j$  and then, they sell the energy to the consumers at prices  $p_i$  and  $p_j$ , respectively.

Taking into account consumer preferences, retailers compete in prices trying to increase their proportion of consumers from the initial values,  $\pi_{ij}$  and  $\pi_{ji}$ , respectively, by the optimization of their utility functions. The following proposition shows the optimal strategy for each retailer to attain this goal.

**PROPOSITION 1.** *The optimal strategy for retailer  $R_i$ , is defined through the solution of the following problem*

$$\max_{p_i} u_i^r(p_i, p_j) = (p_i - c_i)q_i^*(p_i)\pi_i^*(p_i, p_j) \quad (3)$$

where  $q_i^*(p_i) = v(p_i)$  is the optimal consumer energy, and  $\pi_i^*(p_i, p_j)$  is the attained proportion of consumers by retailer  $R_i$  defined as

$$\pi_i^*(p_i, p_j) = F_{ij}^* \pi_{ij} + \bar{F}_{ji}^* \pi_{ji}, \quad (4)$$

where  $F_{ij}^*$  denotes the probability that a consumer who bought from retailer  $R_i$  remains with  $R_i$  and  $\bar{F}_{ji}^*$  the probability that a consumer who bought from retailer  $R_j$  switches to  $R_i$ . These probabilities are obtained as follows

$$F_{ij}^* = P(\theta_{ij} \leq \theta_{ij}^*), \quad \bar{F}_{ji}^* = P(\theta_{ji} > \theta_{ji}^*) \quad (5)$$



where

$$\theta_{ij}^* = \frac{1}{2k} (\varphi(p_i) - \varphi(p_j) + k + \gamma_i). \quad (6)$$

and  $\varphi(p_i) \equiv u(v(p_i)) - pv(p_i)$  is a negative and decreasing function of  $p_i$  that satisfies that  $\varphi'(p_i) = -v(p_i)$ .

*Proof:* From (1) and (2), a consumer who bought from retailer  $R_i$  will remain with  $R_i$  if it holds that

$$u(q_i) - p_i q_i - k\theta_{ij} + \gamma_i \geq u(q_j) - p_j q_j - k\theta_{ji}, \quad (7)$$

or equivalently if

$$\theta_{ij} \leq \theta_{ij}^* = \frac{1}{2k} (\varphi(p_i) - \varphi(p_j) + k + \gamma_i), \quad (8)$$

On the other hand, a consumer who bought from retailer  $R_j$  will switch to  $R_i$  if

$$u(q_i) - p_i q_i - k\theta_{ij} > u(q_j) - p_j q_j - k\theta_{ji} + \gamma_j, \quad (9)$$

or equivalently if

$$\theta_{ji} > \theta_{ji}^* = \frac{1}{2k} (\varphi(p_j) - \varphi(p_i) + k + \gamma_j). \quad (10)$$

□

Due to the symmetry of the problem, a proposition similar to Proposition 1 can be obtained, by interchanging  $i$  and  $j$ , to characterize the optimal strategy of retailer  $R_j$ .

Retailer  $R_i$ 's problem (3) and its equivalent for retailer  $R_j$  are interrelated as both objective functions depend on prices  $p_i$  and  $p_j$ , simultaneously. Under this setting, the following game and associated Nash equilibrium can be defined:

**DEFINITION 1.** (*Nash equilibrium of the duopoly game*) The game between the two retailers  $R_i$  and  $R_j$  is defined when both players seek to determine their optimal  $p_i$  and  $p_j$ , respectively, to maximize their profits (3). A Nash equilibrium would be reached if there exists a set of prices  $p_i^*$  and  $p_j^*$  so that  $u_i^r(p_i^*, p_j^*) \geq u_i^r(p_i, p_j^*) \quad \forall p_i \in S_i$  and if  $u_j^r(p_i^*, p_j^*) \leq u_j^r(p_i^*, p_j) \quad \forall p_j \in S_j$ , where  $S_i$  and  $S_j$  represent the feasible region for prices  $p_i$  and  $p_j$ .

### 3.1. Equilibrium price classification

In this section we classify and characterize the different types of equilibria that could be reached in the market. Some definitions that will be used within this section are introduced below.

**DEFINITION 2.** We say retailer  $R_i$  has a *dominant position* in the Hotelling line if, at equilibrium,  $F_{ij}^* = \bar{F}_{ji}^* = 1$ . Or equivalently, if  $F_{ji}^* = \bar{F}_{ij}^* = 0$ .

Note that in this case all consumers remain with or will switch to the dominant retailer, causing a lack of competition in the market.

DEFINITION 3. We say retailer  $R_i$  has a *leading position* in the Hotelling line if, at equilibrium prices,  $F_{ij}^* = 1$  but  $\bar{F}_{ji}^* < 1$ . Or equivalently, if  $F_{ji}^* < 1$  but  $\bar{F}_{ij}^* = 0$ . In this latter case we say retailer  $R_j$  has a *following position*.

This definition implies all consumers with the leading retailer remain with her, while a proportion of consumers with the follower retailer switch. This causes some competition in the market, with one retailer leading the other one.

DEFINITION 4. We say there is *complete competition* in the Hotelling line if, at equilibrium prices,  $0 < F_{ij}^* < 1$  and  $0 < F_{ji}^* < 1$ .

This definition implies consumers have a certain probability to switch between retailers, allowing more competitive equilibrium prices.

Hereafter, we assume consumer preferences are distributed along the Hotelling line following a uniform distribution, that is,  $F_{ij}(\theta_{ij}) = 0$  for  $\theta_{ij} < 0$ ,  $F_{ij}(\theta_{ij}) = \theta_{ij}$  for  $0 \leq \theta_{ij} \leq 1$ , and  $F_{ij}(\theta_{ij}) = 1$  for  $\theta_{ij} > 1$ .

According to the above definitions and depending on the different market settings, we distinguish three types of equilibrium: dominant position, leading positions, and complete competition.

PROPOSITION 2. *If the following condition holds*

$$\varphi(p_i) - \varphi(p_j) \geq k + \gamma_j, \quad (11)$$

*then retailer  $R_i$  will have a dominant position in the Hotelling line and the optimal price is given by  $(p_i - c_i)v'(p_i) + v(p_i) = 0$ , while retailer  $R_j$  has no demand and gets no income.*

The proof for this proposition can be found in the Appendix.

To state the following propositions, we need to define the following function of prices:

$$\Gamma(p; c) = -\frac{(p - c)v(p)\varphi'(p)}{v(p) + (p - c)v'(p)} - \varphi(p) \quad (12)$$

The conditions under which a retailer will exhibit a leading position in the market are provided in the following proposition.

PROPOSITION 3. *If the following condition holds*

$$\varphi(p_i) - \varphi(p_j) \geq k - \gamma_i, \quad k + \gamma_j > \varphi(p_i) - \varphi(p_j) \geq -k + \gamma_j, \quad (13)$$

*then retailer  $R_i$  will have a leading position in the Hotelling line where the optimal (equilibrium) prices satisfy the following conditions*

$$\Gamma(p_i; c_i) = -\varphi(p_j) + \frac{1 + \pi_{ij}}{\pi_{ji}}k - \gamma_j \quad (14)$$

$$\Gamma(p_j; c_j) = -\varphi(p_i) + k + \gamma_j, \quad (15)$$

**Table 1** Different values of  $\kappa$ 

	$\kappa_i$	$\kappa_j$
Proposition 3	$\frac{1+\pi_{ij}}{\pi_{ji}}k - \gamma_j$	$k + \gamma_j$
Proposition 4	$k + \pi_{ij}\gamma_i - \pi_{ji}\gamma_j$	$k - \pi_{ij}\gamma_i + \pi_{ji}\gamma_j$

The proof for this proposition can be found in the Appendix.

Similarly, Proposition 4 below characterizes the complete competition case.

PROPOSITION 4. *Assuming that  $k \geq (\gamma_i + \gamma_j)/2$ , if the following condition holds*

$$k - \gamma_i \geq \varphi(p_i) - \varphi(p_j) \geq -k + \gamma_j, \quad (16)$$

*then the Hotelling line works under complete competition, where the optimal (equilibrium) prices satisfy the following conditions*

$$\Gamma(p_i; c_i) = -\varphi(p_j) + k + \pi_{ij}\gamma_i - \pi_{ji}\gamma_j \quad (17)$$

$$\Gamma(p_j; c_j) = -\varphi(p_i) + k - \pi_{ij}\gamma_i + \pi_{ji}\gamma_j. \quad (18)$$

The proof for this proposition can be found in the Appendix.

Under the assumption  $k \geq (\gamma_i + \gamma_j)/2$ , the case where both retailers have a leading position ( $F_{ij}^* > 1$  and  $F_{ji}^* > 1$  or equivalently  $\theta_{ij}^* > 1$  and  $\theta_{ji}^* > 1$ ) is not possible, so it is not considered as a potential equilibrium in this work. Additionally, symmetric equilibria can be characterized from propositions 2 and 3 by interchanging retailer  $R_i$  by  $R_j$ .

The equilibrium prices for the preceding competitive cases in Propositions 3 and 4 (we exclude the dominant position from Proposition 2), can be obtained by solving a system of equations of the form

$$\Gamma(p_i; c_i) = -\varphi(p_j) + \kappa_i \quad (19)$$

$$\Gamma(p_j; c_j) = -\varphi(p_i) + \kappa_j, \quad (20)$$

for  $\kappa_i$  and  $\kappa_j$  values depending on  $k$ ,  $\gamma_i$ ,  $\gamma_j$ ,  $\pi_{ij}$  and  $\pi_{ji}$  (but independent of  $p_i$  and  $p_j$ ). Table 1 presents the different values of  $\kappa$ .

In the following, we will exploit this mathematical structure to study the existence and uniqueness of the equilibrium prices.

### 3.2. Existence and uniqueness of equilibrium prices

In this section we characterize the solution of the system (19) and (20), which determines the equilibria solutions for the market configurations where there is some degree of competition, i.e., the cases described within Propositions 3 and 4.

To conduct this study, we will introduce some conditions on the consumers utility function  $u(q)$ . In what follows, we will make use of the notation  $\tilde{c} = \min(c_i, c_j)$ .

We define the following conditions:

C.1 We assume that  $u(0) = 0$  and there exists  $\tilde{q}$  such that  $u'(\tilde{q}) = \tilde{c}$ .

C.2 We assume that  $u$  is three times continuously differentiable on  $[0; \tilde{q}]$ , that it is nondecreasing and concave ( $u''(q) \leq 0$ ) on  $[0; \tilde{q}]$ , and that it has an invertible first derivative  $u'(q) \geq 0$  on that interval.

C.3 We impose the following condition on  $u'''$ ,

$$\frac{u'''(v(p))}{u''(v(p))^2} \leq \frac{1}{p - \tilde{c}} - \frac{1}{v(p)u''(v(p))}.$$

Conditions C.1 and C.2 are standard. A first interpretation of condition C.3 is that the function  $u$  should not change curvature too fast. Note that a sufficient condition for C.3 is  $u'''(q) \leq 0$ ; also, if the function  $u$  satisfies condition C.2 on  $[0, \infty)$  and the sign of  $u'''(q)$  is constant for all large  $q$ , then it must hold  $u'''(q) \leq 0$  for large  $q$ .

Our first result is related to the function  $\Gamma$  defined above, which using  $\varphi'(p) = -v(p)$  can be written as

$$\Gamma(p; c) \equiv \frac{(p - c)v(p)^2}{v(p) + (p - c)v'(p)} - \varphi(p). \quad (21)$$

The following two lemmas characterize key analytical properties of  $\Gamma(p; c)$  that are needed to prove the existence of equilibrium.

LEMMA 1. For  $p \in [c; \alpha]$ ,  $\Gamma(p; \alpha, c)$  defined in (21) has at least one vertical asymptote.

*Proof:* This result follows from the continuity of  $d(p; c) = v(p) + (p - c)v'(p)$  and the properties that  $d(c; c) = v(c) > 0$  and  $d(\alpha; c) = v(\alpha) + (\alpha - c)v'(\alpha) = (\alpha - c)v'(\alpha) < 0$ , as  $v(\alpha) = 0$  and  $v'(\alpha) = 1/u''(v(\alpha)) = 1/u''(0) < 0$  from Condition C.2.  $\square$

In what follows we will use the notation

$$\tau(c) \equiv \min \{ \omega : d(\omega; c) = 0, \omega \in [c; \alpha] \},$$

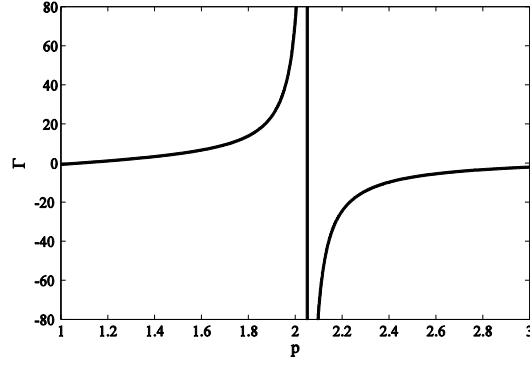
for the first pole of  $\Gamma$  on  $[c; \alpha]$ .

An illustration of a plot for a particular realization of function  $\Gamma$  can be seen in Figure 1.

LEMMA 2. For  $p \in [c; \tau(c))$ ,  $\Gamma(p; c)$  defined in (21) is a continuous monotonically increasing function taking values in  $[-\varphi(c); \infty)$ .

*Proof:* The derivative of  $\Gamma$  can be written as

$$\Gamma'(p; c) = v + \frac{v(p - c)}{(v + (p - c)v')^2} (-vv' + (p - c)(v')^2 - (p - c)vv''). \quad (22)$$



**Figure 1** A particular realization of the  $\Gamma$  function.

From Condition C.2, on  $[c; \tau)$  we have that  $v$  has continuous first and second derivatives, with  $v(p) > 0$  and  $v'(p) < 0$ . Also, as  $v'(p) = 1/u''(v(p))$  and  $v''(p) = -u'''(v(p))v'(p)/(u''(v(p)))^2$ , Condition C.3 implies

$$(p - c)v(p)v''(p) \leq -vv' + (p - c)(v')^2,$$

and as a consequence  $\Gamma'(p; c) > 0$  on  $[c; \tau)$ .  $\square$

The following result proves the existence of solutions for the system of equations of interest,

$$\Gamma(p_i; c_i) = -\varphi(p_j) + \kappa_i \tag{23}$$

$$\Gamma(p_j; c_j) = -\varphi(p_i) + \kappa_j, \tag{24}$$

for  $\kappa_i$  and  $\kappa_j$  defined in Table 1, under the following condition:

C.4 The values  $\kappa_i$  satisfy

$$\kappa_i \geq \varphi(c_j) - \varphi(c_i), \quad \kappa_j \geq \varphi(c_i) - \varphi(c_j).$$

In other words, Theorem 1 below proves the existence of equilibrium solutions for the retailers pricing game for two cases: i) when one of the retailers has a leading position in the market (Proposition 3) and ii) when there is complete competition (Proposition 4). This result completes all the possible market configurations, as the dominant retailer case was already characterized in Proposition 2.

**THEOREM 1.** *Under Conditions C.1–C.4, there exists at least a solution for the system (23)–(24). Furthermore, this solution is in  $[c_i; \tau(c_i)) \times [c_j; \tau(c_j))$ .*

The proof for this theorem can be found in the Appendix.

The next step is to establish the uniqueness of the preceding fixed point. To prove this we need to start by tightening Condition C.3.

Define

$$\rho(p; c) \equiv \frac{p - c}{(v + (p - c)v')^2} \left( -vv' + (p - c)(v')^2 - (p - c)vv'' \right). \quad (25)$$

Also, let

$$\begin{aligned} \bar{c}_i &\equiv G(-\varphi(c_i) + \kappa_i; c_i) \geq c_i \\ \bar{c}_j &\equiv G(-\varphi(c_j) + \kappa_j; c_j) \geq c_j. \end{aligned}$$

These values have the property that the iterates generated by the fixed-point equations (EC.7)–(EC.8) lie in  $[\bar{c}_k; \tau(c_k))$  for  $k = i, j$ . We also define  $\bar{c} \equiv \min(\bar{c}_i, \bar{c}_j)$  and  $\bar{\tau} \equiv \max(\tau(c_i), \tau(c_j))$ .

Condition C.3 is replaced with

C.3' The values of  $u$ ,  $c_k$  and  $\kappa_k$  for  $k = i, j$  are such that for all  $p \in [\bar{c}_k; \tau(c_k))$ ,  $k = i, j$ , it holds that

$$1 + \rho(p; c_k) > \sqrt{\frac{v(\bar{c})}{v(\bar{\tau})}}.$$

As condition C.3 is equivalent to requiring

$$\rho(p; c_k) \geq 0,$$

and  $v(\bar{c}) > v(\bar{\tau})$ , condition C.3' can be interpreted as a tightening of C.3.

Another motivation to introduce the  $\bar{c}_k$  values is that while it holds that  $\rho(c_k; c_k) = 0$ , we have  $\rho(\bar{c}_k; c_k) \geq 0$ .

The following theorem establishes the uniqueness of the equilibrium solutions for the retailers pricing game: i) the leading retailer case (Proposition 3) and ii) for the complete competition case (Proposition 4).

**THEOREM 2.** *Under Conditions C.1, C.2, C.3' and C.4, the solution of (23)–(24) on  $[\bar{c}_i; \tau(c_i)) \times [\bar{c}_j; \tau(c_j))$  is unique.*

The proof for this theorem can be found in the Appendix.

### 3.3. Sensitivity analysis

In this section we provide some guidelines on how to compute sensitivities of the equilibrium prices with respect to the problem parameters. To this end, consider our fixed-point iteration (23)–(24), written in the form

$$\Gamma(p_i; c_i) + \varphi(p_j) = \kappa_i \quad (26)$$

$$\Gamma(p_j; c_j) + \varphi(p_i) = \kappa_j, \quad (27)$$

for  $\kappa_i$  and  $\kappa_j$  given by (1).

Let  $\nu$  denote any of the parameters  $k$ ,  $\gamma_i$ ,  $\gamma_j$ ,  $\pi_{ij}$ ,  $\pi_{ji}$ . The sensitivity of the equilibrium prices with respect to  $\nu$  can be found from the solution of

$$\begin{aligned}\Gamma'(p_i; c_i) \frac{\partial p_i}{\partial \nu} + \varphi'(p_j) \frac{\partial p_j}{\partial \nu} &= \frac{\partial \kappa_i}{\partial \nu} \\ \Gamma'(p_j; c_j) \frac{\partial p_j}{\partial \nu} + \varphi'(p_i) \frac{\partial p_i}{\partial \nu} &= \frac{\partial \kappa_j}{\partial \nu}.\end{aligned}$$

Using (EC.12), we can rewrite these equations as

$$\begin{aligned}(1 + \rho(p_i; c_i)) \frac{\partial p_i}{\partial \nu} - \frac{v(p_j)}{v(p_i)} \frac{\partial p_j}{\partial \nu} &= \frac{1}{v(p_i)} \frac{\partial \kappa_i}{\partial \nu} \\ -\frac{v(p_i)}{v(p_j)} \frac{\partial p_i}{\partial \nu} + (1 + \rho(p_j; c_j)) \frac{\partial p_j}{\partial \nu} &= \frac{1}{v(p_j)} \frac{\partial \kappa_j}{\partial \nu}.\end{aligned}$$

From (22), under Condition C.3 we have that  $\rho_k \geq 0$  ( $k = i, j$ ), and under mild additional conditions (also under Condition C.3', for example),  $\rho_k > 0$  and the coefficient matrix in the preceding system is invertible. The values of interest are given by

$$\begin{aligned}v(p_i) \Delta_{ij} \frac{\partial p_i}{\partial \nu} &= \rho(p_j; c_j) \frac{\partial \kappa_i}{\partial \nu} + \frac{\partial \kappa_i}{\partial \nu} + \frac{\partial \kappa_j}{\partial \nu} \\ v(p_j) \Delta_{ij} \frac{\partial p_j}{\partial \nu} &= \rho(p_i; c_i) \frac{\partial \kappa_j}{\partial \nu} + \frac{\partial \kappa_i}{\partial \nu} + \frac{\partial \kappa_j}{\partial \nu},\end{aligned}$$

where  $\Delta_{ij} = \rho(p_i; c_i) + \rho(p_j; c_j) + \rho(p_i; c_i)\rho(p_j; c_j)$ .

For example, for  $\nu \equiv \pi_{ij}$  and from (1) it will hold that

	$\partial \kappa_i / \partial \nu$	$\partial \kappa_j / \partial \nu$
Proposition 3	$\frac{1}{\pi_j} k$	0
Proposition 4	$\gamma_i$	$-\gamma_i$

and for the case in Proposition 4:

$$\begin{aligned}v(p_i) \Delta_{ij} \frac{\partial p_i}{\partial \pi_i} &= \rho(p_j; c_j) \gamma_i \\ v(p_j) \Delta_{ij} \frac{\partial p_j}{\partial \pi_j} &= -\rho(p_i; c_i) \gamma_i,\end{aligned}$$

implying in particular  $\partial p_i / \partial \pi_{ij} > 0$  and  $\partial p_j / \partial \pi_{ij} < 0$ . The higher the initial proportion of consumers that a retailer has, the higher the prices he can set at the equilibrium.

By following a similar reasoning we can compute the signs associated to the rest of price sensitivities, for the two market setting where competition is present:

	$\frac{\partial p_i}{\partial \pi_{ij}}$	$\frac{\partial p_i}{\partial \pi_{ji}}$	$\frac{\partial p_j}{\partial \pi_{ij}}$	$\frac{\partial p_j}{\partial \pi_{ji}}$	$\frac{\partial p_i}{\partial k}$	$\frac{\partial p_j}{\partial k}$	$\frac{\partial p_i}{\partial \gamma_i}$	$\frac{\partial p_i}{\partial \gamma_j}$	$\frac{\partial p_j}{\partial \gamma_i}$	$\frac{\partial p_j}{\partial \gamma_j}$
Proposition 3	$> 0$	$< 0$	$> 0$	$< 0$	$> 0$	$> 0$	$= 0$	$< 0$	$= 0$	$> 0$
Proposition 4	$> 0$	$< 0$	$< 0$	$> 0$	$> 0$	$> 0$	$> 0$	$< 0$	$< 0$	$> 0$

We obtain that the transportation costs  $k$  always have a positive relationship with the equilibrium prices. For the competitive case in Proposition 4, an increase of  $\gamma_i$  leads to an increment on the

price offered by retailer  $R_i$  while decreasing the price offered by the competitor  $R_j$ . However, for the market described in Proposition 3, as  $R_i$  is the leader, its  $\gamma_i$  does not have a direct impact on prices. Additionally, it should be noted that, for the case described in Proposition 3,  $\partial p_i / \partial \pi_{ij} > 0$  and  $\partial p_j / \partial \pi_{ij} > 0$ , indicating that an increase in the initial market share of the leader gives rise to simultaneous increases for both the leader's and follower's price, as the latter cannot capture any of the leader's consumers (it lacks any incentive to decrease prices).

These and other relevant sensitivities will be further studied in Section 5 for a particular realization of the consumers' utility function  $u(q)$ .

#### 4. Electricity market with more than two retailers: network

In this section we extend the previous duopoly model to include additional retailers. In particular, we assume that retailers are located at the nodes of a pairwise network while consumers are distributed within the Hotelling lines connecting these nodes. Similar to Somaini and Einav (2013), we consider that each consumer can only buy electricity from the two retailers located at the two ends of her corresponding line. We adopt this simplification for the sake of clarity and to ease the computational burden of the model. Nevertheless, this framework is very similar to the actual retail market in many countries where, despite the existence of many retailing companies, these are distributed within the electricity network so that consumers have only a small set of options to sign contracts with.

We have a network with a set of retailers,  $\mathcal{N} = \{1, 2, \dots, N\}$ , and a set of lines,  $\mathcal{L} = \{1, 2, \dots, L\}$ . For each line  $ij \in \mathcal{L}$  we have different types of consumers with preferences  $\theta_{ij}$  over the corresponding Hotelling line. Because consumers in the same line are equivalent, we have

$$\theta_{ij} + \theta_{ji} = 1. \quad (28)$$

Each retailer purchases their energy from the spot or futures market, at a cost  $c_i$ , for  $i = 1, \dots, N$ . Then, they sell the energy to all the consumers at a unique price  $p_i$ .

The consumers are shared through the network with initial proportions  $\pi_{ij}^0 = \pi_{ji}^0$  for each line  $ij \in \mathcal{L}$ , with  $\sum_{i,j>i} \pi_{ij}^0 = 1$ . And for each line, let  $\pi_{ij}$  and  $\pi_{ji} = 1 - \pi_{ij}$  denote the initial proportion of consumers within that line with  $R_i$  and  $R_j$ , respectively.

The following proposition shows the optimal strategy for each retailer to increase its proportion of consumers.

**PROPOSITION 5.** *The optimal strategy for retailer  $R_i$ , for  $i = 1, \dots, N$ , is defined through the solution of the following problem*

$$\max_{p_i} u_i^r(p_i, p_{-i}) = (p_i - c_i) q_i^*(p_i) \pi_i^*(p_i, p_{-i}) \quad (29)$$



where  $q_i^*(p_i) = v(p_i)$  is the optimal consumer energy, and  $\pi_i^*(p_i, p_{-i})$  is the attained proportion of consumers by retailer  $R_i$  defined as

$$\pi_i^*(p_i, p_{-i}) = \sum_{j \in \mathcal{N}_i} \pi_{ij}^*(p_i, p_{-i}) \pi_{ij}^0 \quad (30)$$

where  $\mathcal{N}_i$  denotes the set of retailers adjacent to  $R_i$  and

$$\pi_{ij}^*(p_i, p_{-i}) = F_{ij}^* \pi_{ij} + \bar{F}_{ji}^* \pi_{ji} \quad (31)$$

is the proportion of consumers in line  $ij$  that buy from  $R_i$  where  $F_{ij}^*$  and  $\bar{F}_{ji}^*$  are defined in Proposition 1.

The proof is equivalent to that for Proposition 1.

The analytical characterization of the equilibrium prices of the multiple retailer framework is more complex than in the duopoly case. Hence, we will limit our analysis to a numerical case study. The equilibrium prices will be computed by solving simultaneously the optimality conditions for all the retailers, i.e., we solve the following nonlinear system of equations

$$\frac{\partial u_i^r(p_i, p_{-i})}{\partial p_i} = 0 \quad \forall i = 1, \dots, N. \quad (32)$$

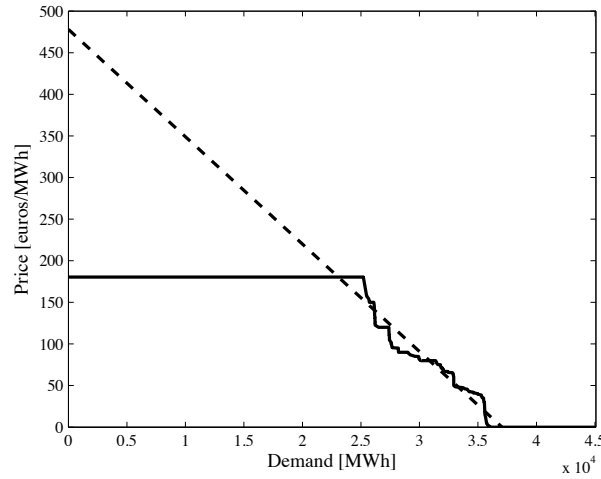
To make sure that the above system provides equilibrium prices, an iteration of a diagonalization solution algorithm is performed, where retailers solve iteratively their profit maximization problems assuming that the rival prices are fixed. If no retailer deviates unilaterally from their price strategy, then the resulting set of prices can be considered as a Nash equilibrium. Additionally, for the sake of clarity we will only present results for market settings in which all the retailers have a competitive position in the market (a case analogous to Proposition 4).

## 5. Empirical analysis

In this section we perform a set of numerical analyses to better understand the impact of different model parameters on the equilibrium market outcomes. First, we focus our analysis on the duopolistic case and then we extend it to the multiple-retailers case.

### 5.1. Duopoly case

**5.1.1. Data** In this case study we use a particular realization of the consumers' utility function  $u(q)$  in (1). In particular, we have adopted a quadratic formulation, which is an standard assumption within the electricity markets literature Oliveira et al. (2013), Kirschen (2003). Hence, we consider that  $u(q) = \alpha q - \frac{1}{2} \beta q^2$ . This is a concave function for  $\beta \geq 0$  which meets the existence and uniqueness equilibrium conditions discussed in Theorem 1 and, for appropriate values of the parameters, also those of Theorem 2.



**Figure 2** Demand curve Spanish day-ahead electricity market (12:00, 04/08/2015) and its linear approximation

To obtain realistic market outcomes, we have adjusted the demand parameters by using data from a real world electricity market: the Spanish electricity market OMIE (2015). Fig. 2 represents the day-ahead aggregated demand for the Spanish market for a selected day and hour. The demand curve can be identified as the consumers' marginal utility, which for our case is a linear curve, i.e.,  $P = \frac{\partial u(q)}{\partial q} = \alpha - \beta q$ . Hence, we approximate the elastic part of the demand by this marginal utility (dash line) which renders  $\alpha = 480$  euros/MWh and  $\beta = 0.012$  euros/MWh<sup>2</sup>. Observe that to make this approximation we have assumed that the Spanish demand function is directly submitted by the consumers, without intermediate retailers. This is not completely true in the Spanish case as a significant part of the day-ahead energy is traded by retailers in the market. However, retailers are supposed to transfer the final consumers' demand to the market (by including an small revenue margin) so that the main behavior of the consumers demand can be assumed to be that of Fig. 2.

For the base case we assume that retailer  $R_i$  is able to purchase its energy in the futures or day-ahead market at a overall cheaper price than retailer  $R_j$  so that  $c_i = 40$  euros/MWh and  $c_j = 47$  euros/MWh. Furthermore, we fix the initial market share of each retailer at 50%, i.e.,  $\pi_{ij} = 1 - \pi_{ji} = 0.5$ .

The selection of adequate values for the transportation costs  $k$  and the loyalty rewards  $\gamma_i$  and  $\gamma_j$  is not straightforward, as there is not much real data available for electricity markets (Waterson 2003). Therefore, the values for  $k$ ,  $\gamma_i$  and  $\gamma_j$ , have been selected so that: a) they have a significant impact on the consumers utility functions (1) and (2) and b) they meet the condition  $k \geq (\gamma_i + \gamma_j)/2$ , which was required by propositions 3 and 4 to avoid unrealistic equilibria. We assume that  $k = 150000$  euros,  $\gamma_i = 70000$  euros and  $\gamma_j = 30000$  euros. Observe that retailer  $R_i$  has a better position in the market as it presents lower costs  $c_i$  and higher loyalty premiums  $\gamma_i$  than  $R_j$ .

**5.1.2. Impact of  $\pi_{ij}$**  First we analyze the impact of the initial market share of retailer  $R_i$  ( $\pi_{ij}$ ) on the market equilibrium outcomes. Results are presented in Fig. 3. For the range  $\pi_{ij} \in (0, 0.385)$ ,  $R_i$  has a leading position in the market (Proposition 3) while for  $\pi_{ij} \geq 0.385$  there is complete competition (Proposition 4). This is a counterintuitive result that indicates that a retailer may lose market power if it increases its initial market share.

This behavior can be explained by analyzing the prices offered by each retailer. If the prices offered by the leader  $R_i$  are too high Fig. 3(a), consumers, initially attached to the leader, will start to consider again the possibility to switch to  $R_j$ , opening competition and altering the market equilibrium. In general, as  $R_i$  has lower costs, it is able to offer lower prices than  $R_j$  for both the leader and complete competition cases. In particular, for the leader case ( $\pi_{ij} \in (0, 0.385)$ ) an increase of  $\pi_{ij}$  increases both  $p_i$  and  $p_j$  dramatically. However, if the market enters into complete competition, there is a discrete negative jump on both prices and only the leader's price  $p_i$  increases again with  $\pi_{ij}$ . Note that these price trends are coherent with the sensitivity analysis performed in Section 3.3.

The resulting market share for each retailer is presented in Fig. 3(b). The appreciable increase on prices in the leader's case entails a small decrease on the leader's market share which is corrected as the market enters complete competition. However, observe that retailer  $R_i$ 's profit, specially when she acts as a leader, always increases with its initial market share (Fig 3(c)), although there is a discrete decrease when the market enters complete competition. Despite its initial market share decrease, retailer  $R_j$ 's profit also increases in the leader case, as she indirectly benefits from the rise of prices.

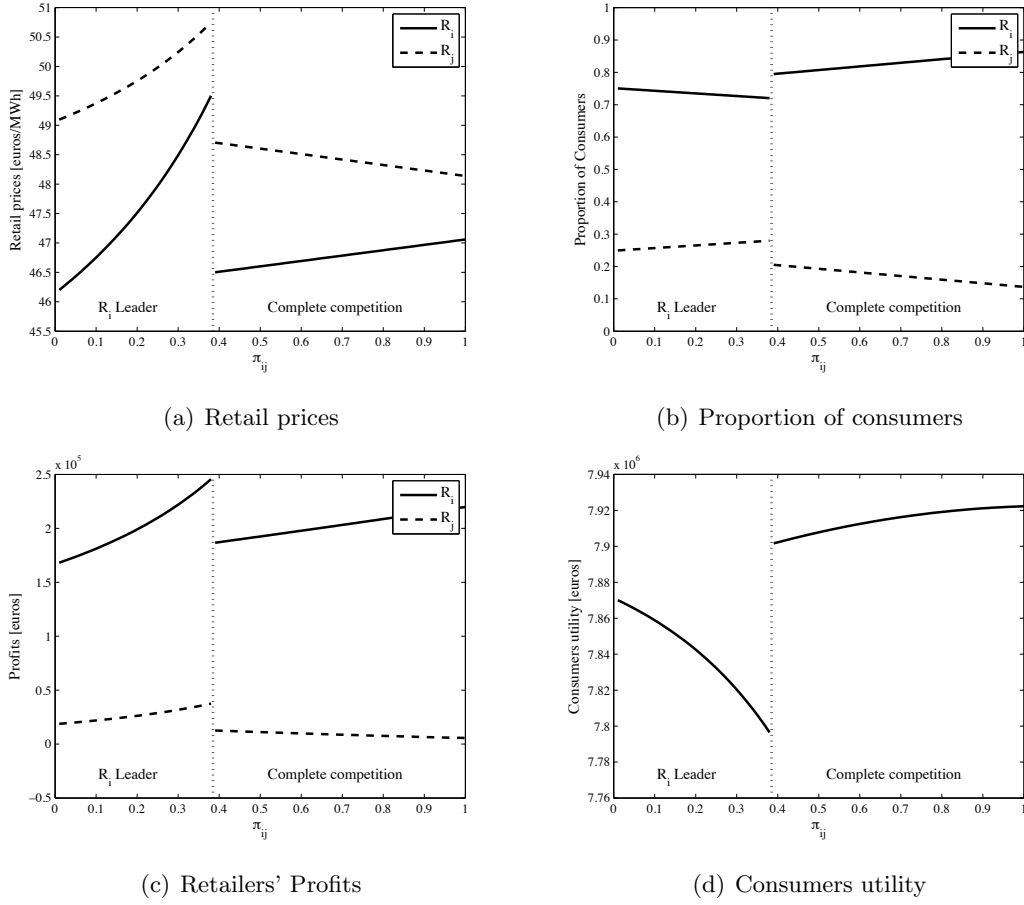
Finally, Fig. 3(d)) presents the consumers utility, which decreases as the leader increases its initial market share but increases again when the market enters complete competition.

**5.1.3. Impact of  $\gamma_i$**  The impact of the loyalty reward  $\gamma_i$  of retailer  $R_i$  on the different market outcomes is presented in Fig. 4.

The market is in complete competition if  $\gamma_i \in (0, 71000)$  euros, and  $R_i$  becomes the leader for values  $\gamma_i \geq 71000$  that imply that is able to "lock-in" its consumers. Observe that once  $\gamma_i \geq 71000$  the loyalty reward  $\gamma_i$  no longer has an impact on the market.

Regarding the retail prices (Fig. 4(a)), an increase of  $\gamma_i$  allows  $R_i$  to charge higher prices to its consumers while  $R_j$  needs to reduce its price to be more competitive. Again, once the market leaves complete competition there is a discrete increase for both prices.

Similarly, for the complete competition case, the market share (Fig. 4(a)) of  $R_i$  increases with its loyalty reward while it decreases for  $R_j$ . For the leading case  $R_i$ 's market share drops slightly to 0.7, as the price offered (Fig. 4(a)) for this range of values of  $\gamma_i$  is very high. The overall effect



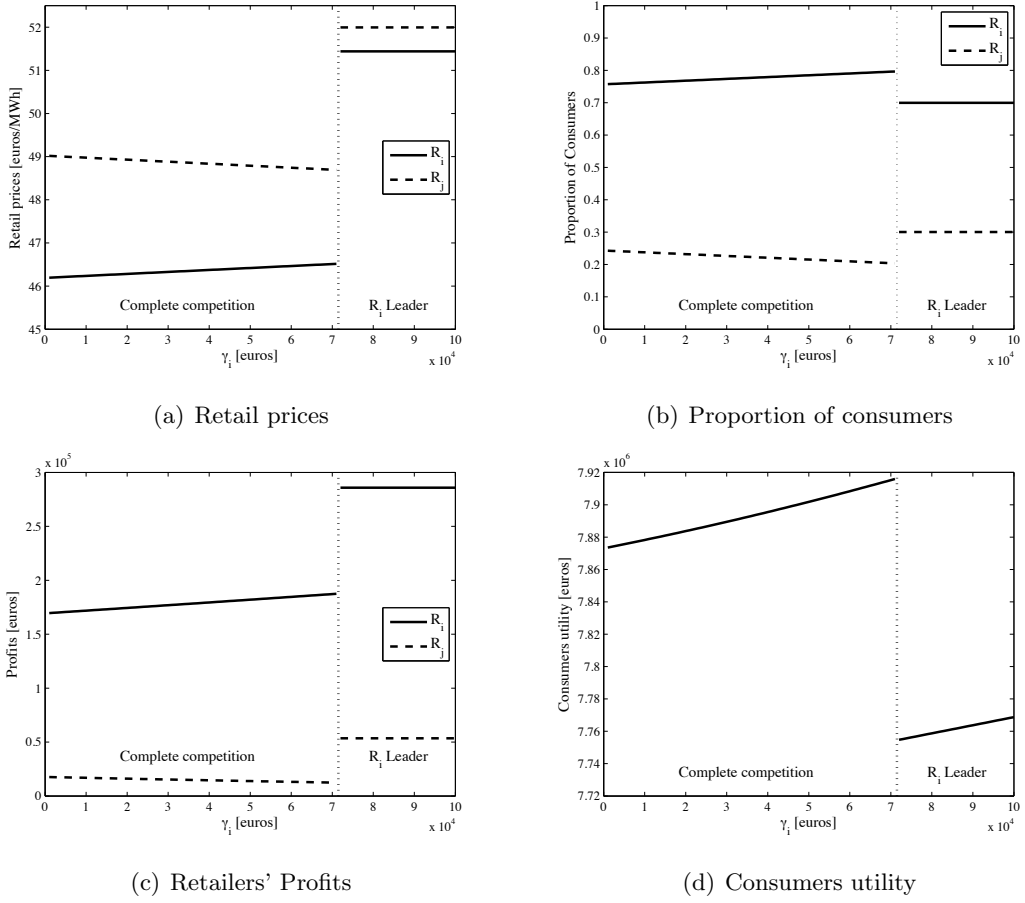
**Figure 3** Impact of  $\pi_{ij}$  on the equilibrium market outcomes

is that  $R_i$ 's profit significantly increases with  $\gamma_i$  (Fig. 4(c)). Observe that  $R_j$  also benefits from the overall higher prices in the leading case by increasing its profit.

Finally, consumers' utility increases with  $\gamma_i$  as more consumers stay with  $R_i$ , which is the cheaper producer. However, consumers utility drops significantly when the market leaves complete competition.

**5.1.4. Impact of  $c_i$**  Fig. 5 presents the impact of the retailer  $R_i$ 's cost on the market outcomes, where we assume that both retailers offer the same loyalty rewards:  $\gamma_i = \gamma_j = 50000$  euros. When retailer  $R_i$ 's costs  $c_i$  are sufficiently low,  $c_i \leq 38.8$  euros/MWh (remember that  $R_j$  cost is  $c_j = 47$  euros/MWh), then it can act as a leader. On the contrary, when  $c_i \geq 38.8$  euros/MWh the market enters complete competition.

Both retail prices increase with  $c_i$ , although they drop when the market enters complete competition. When  $c_i = c_j = 47$  euros/MWh both retailers offer the same price as their positions are symmetric.



**Figure 4** Impact of  $\gamma_i$  on the equilibrium market outcomes.

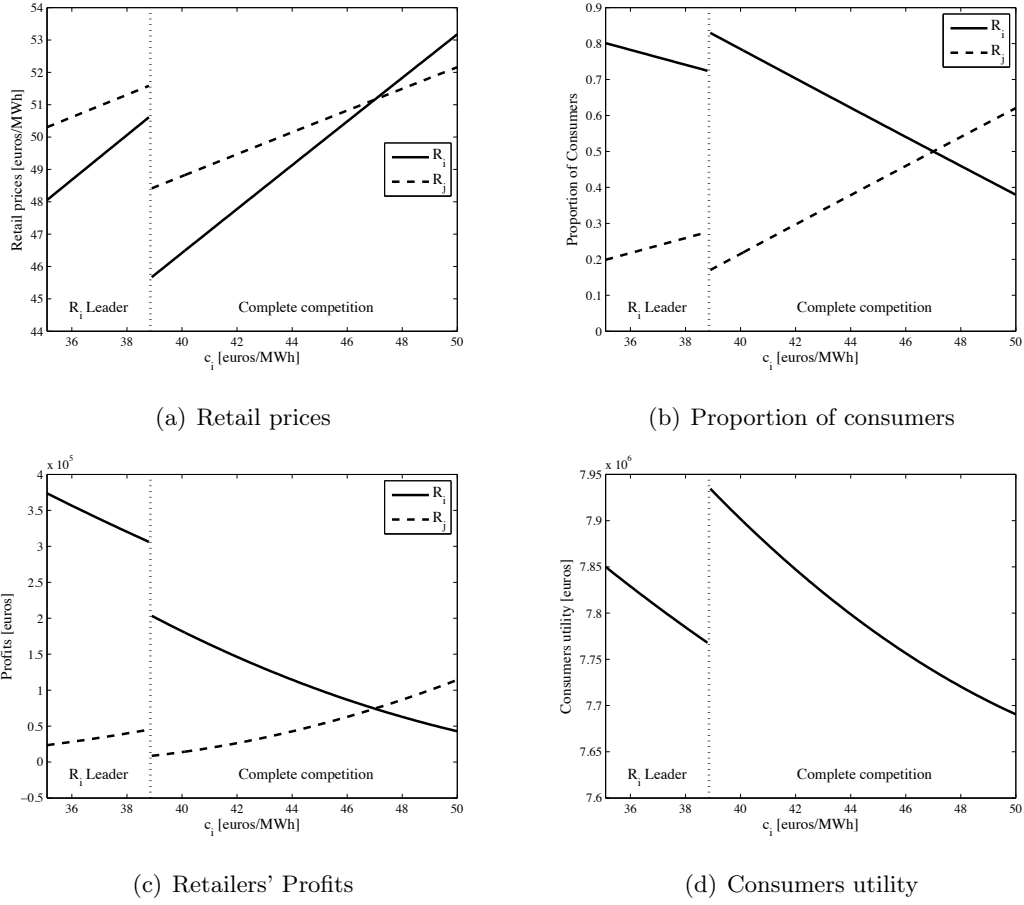
The increase of  $c_i$  implies a decrease in its market share (Fig. 5(b)) and its profit (Fig. 5(c)) as it becomes less competitive with respect to  $R_j$ , which simultaneously increases its market share and its profit.

Finally, an increase in the retailer costs always decreases the consumers' utility (Fig. 5(d)). However, if there is a leader, an small increase in its costs may have a beneficial effect as the market may enter complete competition.

## 5.2. Multiple-retailers

In this section we extend the numerical analysis to the multiple retailers case presented in Section 4. In what follows, we will study the impact of the cost  $c_i$  and the loyalty incentive  $\gamma_i$  on the market equilibrium outcomes, under different network configurations.

**5.2.1. Data** For the base case, we consider four retailers ( $R_1$ ,  $R_2$ ,  $R_3$  and  $R_4$ ) that supply energy to the final consumers. We assume that the retailer costs are  $c_1 = 40$ ,  $c_2 = 42$ ,  $c_3 = 45$  and

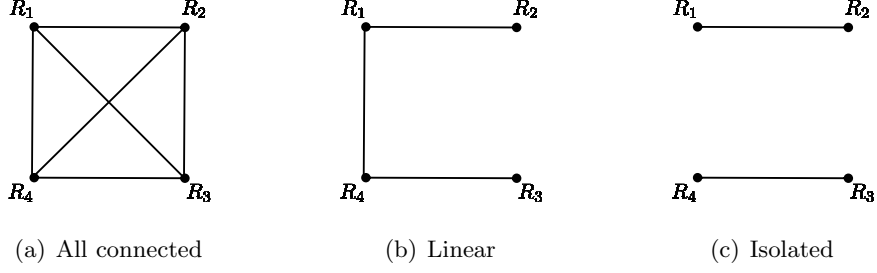


**Figure 5** Impact of  $c_i$  on the equilibrium market outcomes.

$c_4 = 48$  euros/MWh. Furthermore, we assume symmetric loyalty premiums so that  $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 50000$  euros, and transportation costs  $k = 150000$  euros.

We analyze the market equilibrium under the three network configurations shown in Fig. 6. Note that a connection between  $R_i$  and  $R_j$  represents a Hotelling line so that consumers located in that line can buy their energy only from  $R_i$  or  $R_j$ . In the first case depicted in Fig. 6(a), there are connections between all retailers implying that there exists one group of consumers for each combination of any two retailers. The second case considers a linear network (Fig. 6(a)) where there are three groups of consumers that can buy from  $R_1$  or  $R_2$ , from  $R_1$  or  $R_4$  and from  $R_3$  or  $R_4$ , respectively. Finally, the third case assumes that there are two isolated groups of retailers and consumers. The first group of consumers can select between  $R_1$  and  $R_2$  while the second one selects between  $R_3$  and  $R_4$ .

In all these three cases, we assume that consumers are initially uniformly distributed along the existing lines and, for each line, the initial proportion of consumers that bought from each retailer is 0.5. More specifically, for the first case:  $\pi_{ij}^0 = 1/6$  and  $\pi_{ij} = 1/2$  for  $i = 1, 2, 3, 4$ ,  $j = 1, 2, 3, 4$  and



**Figure 6** Network configurations

$i \neq j$ . For the second case:  $\pi_{ij}^0 = 1/3$  and  $\pi_{ij} = 1/2$  for  $(i, j) = (\{1, 2\}, \{1, 4\}, \{3, 4\})$ . For the third case:  $\pi_{ij}^0 = 1/2$  and  $\pi_{ij} = 1/2$  for  $(i, j) = (\{1, 2\}, \{3, 4\})$ .

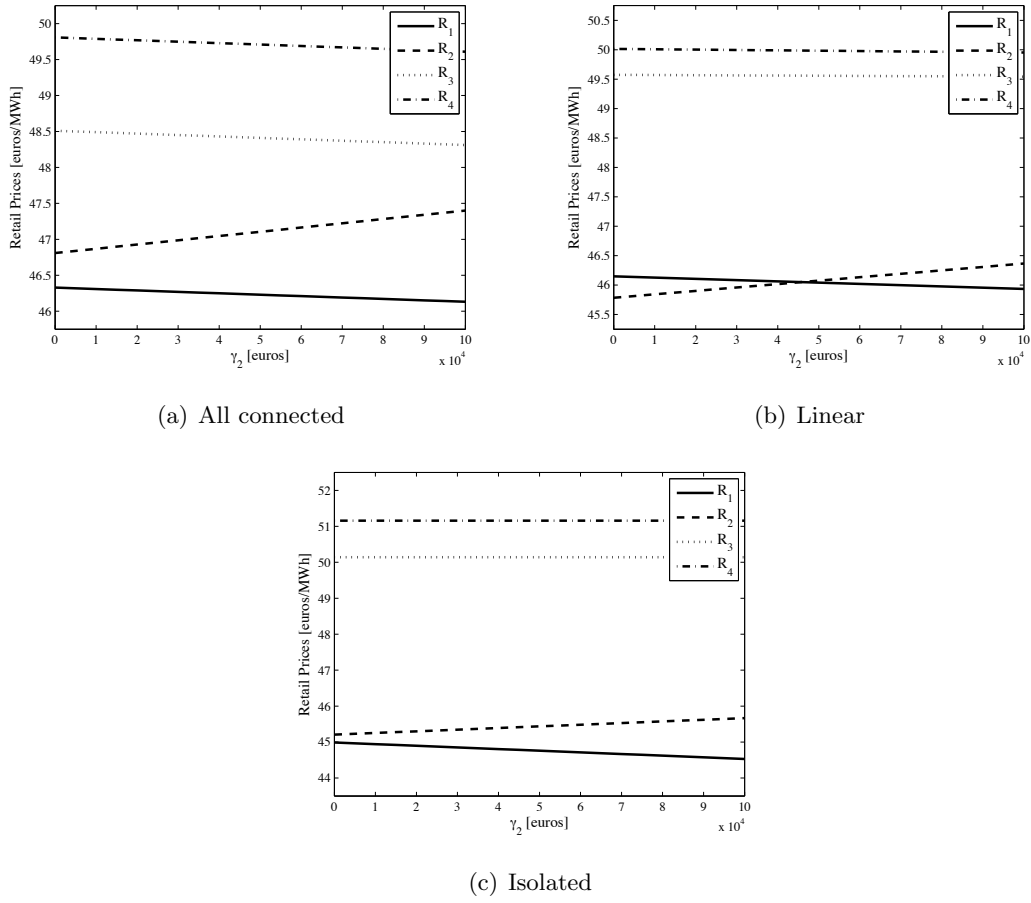
For the sake of clarity, all the solutions presented within the following sections correspond to market equilibria with complete competition, i.e., no retailer acts as a leader in any of the Hotelling lines.

**5.2.2. Impact of  $\gamma_i$**  In this section we analyze the impact of the value of retailer  $R_2$ 's loyalty reward  $\gamma_2$  on some market outcomes. Fig. 7 represents how retail prices evolve with  $\gamma_2$  for the three network configurations shown in Fig. 6. As observed in the duopoly case, the retail price offered by  $R_2$  increases with its loyalty reward while the prices offered by its rivals decrease. This influence is stronger for the case in which all of the retailers are connected (Fig. 7(a)) and becomes less relevant as the network gets less connected (Fig. 7(c)). Similarly, the difference between the retailer prices (maximum minus minimum price) increases as consumers are more isolated.

A similar behavior is observed when analyzing the market shares for each retailer (Fig. 8). Retailer  $R_2$  increases its market share with  $\gamma_2$  while decreasing those of its competitors. It is relevant to notice that the linear network configuration is specially favorable to retailer  $R_1$  as it supplies energy to two groups of consumers, while being the cheapest retailer in both Hotelling lines. On the contrary, retailer  $R_4$ , that is also present in two Hotelling lines, cannot attain a significant market share as it offers the highest prices.

The profits for each retailer and network configuration are presented in Fig. 9. Retailer  $R_2$ 's profits increase with greater values of its loyalty reward  $\gamma_2$ . The rest of retailers decrease their profit, although this effect disappears for retailers  $R_3$  and  $R_4$  if they are isolated from retailer  $R_2$  (Fig. 9(c)). Again, we can observe that the linear network configuration is beneficial for retailer  $R_1$  as it achieves significantly greater profits than its rivals.

The consumers' utility is presented in Fig. 10. For the three cases considered, the utility increases with  $\gamma_2$  where the worst configuration is the isolated network. However, for small values of  $\gamma_2$  the linear case yields higher utility values, while for bigger values of  $\gamma_2$  the all-connected case offers better results. This can be explained by noticing that in the linear case, half of the consumers are



**Figure 7** Impact of  $\gamma_2$  on equilibrium prices.

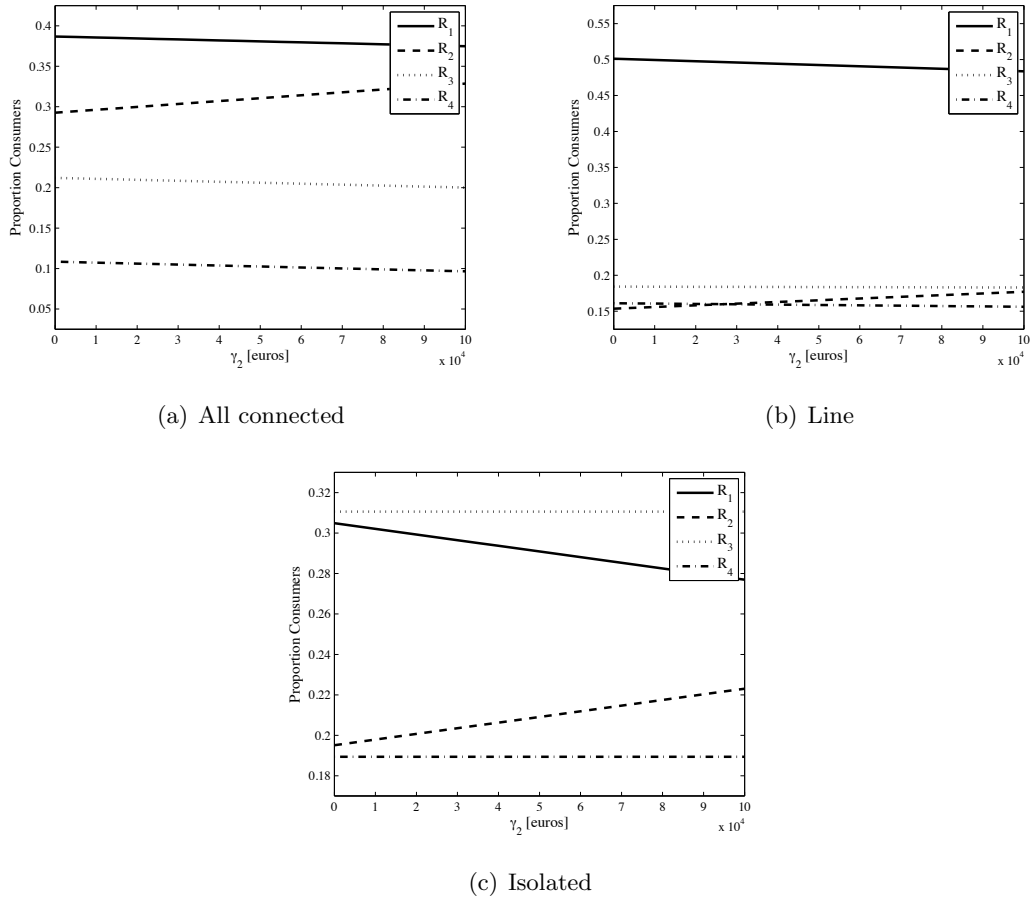
with retailer  $R_1$ , which offers the cheapest prices. Hence, if  $\gamma_2$  is not sufficiently large, this case would be preferable for consumers. However,  $R_2$  has a greater market share in the all-connected configuration so that further increasing its  $\gamma_2$  would yield a higher impact on consumers' utilities.

**5.2.3. Impact of  $c_i$**  Now we study how the cost increment of one retailer, in particular retailer  $R_2$ 's cost  $c_2$ , affects prices, market shares and profits.

Fig. 11 depicts the retail prices for different network configurations. An increment in retailer  $R_2$  cost yields higher prices for all retailers, although this increment is bigger for  $R_2$ . Again, a more interconnected network (Fig. 11(a)) implies more homogeneous retail prices, but at the same time prices are more sensitive to variations in a single retailer costs. For the isolated network (Fig. 11(c)), retailers  $R_3$  and  $R_4$  are not affected by  $c_2$ .

Fig. 12 presents the retailers' market shares. The cost increment of  $R_2$  makes it less competitive, thus reducing its market share while increasing its rivals' ones. This effect is mitigated for retailers  $R_3$  and  $R_4$  for the isolated case (Fig. 12(c)).





**Figure 8** Impact of  $\gamma_2$  on equilibrium market shares.

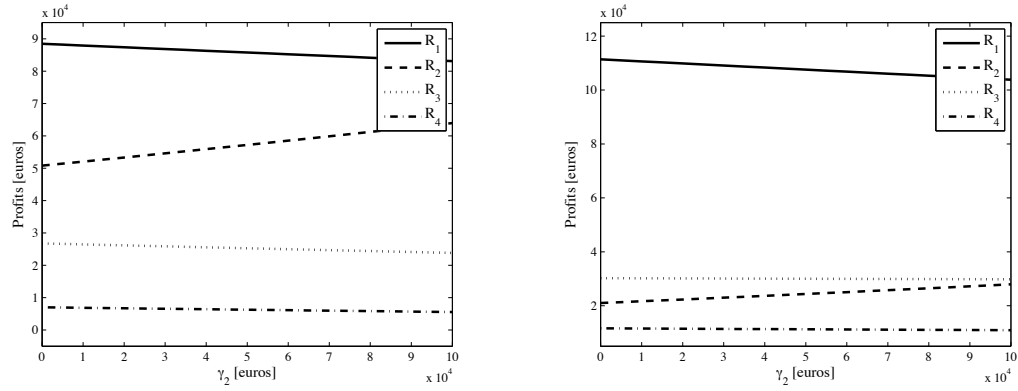
As Fig. 13 shows, the trend for profits is similar to the trend for market shares. Retailer  $R_2$ 's profits decrease with its costs, while rival retailers benefit from the reduction in  $R_2$ 's competitiveness. In particular, retailer  $R_1$ , which has the lowest costs, also presents the largest profit increases as it is always connected to  $R_2$ .

Finally, consumer utilities decrease with the increment of retailer  $R_2$  costs. As it was observed before, the worst network configuration is when  $R_1$  and  $R_2$  are isolated from  $R_3$  and  $R_4$ . For small values of  $c_2$ , the maximum utility is achieved for the all-connected network. However, as  $c_2$  increases the linear configuration is preferable, since the high prices offered by  $R_2$ , which is located at one extreme of the network, find it more difficult to have an effect on the other retailers.

## 6. Conclusions

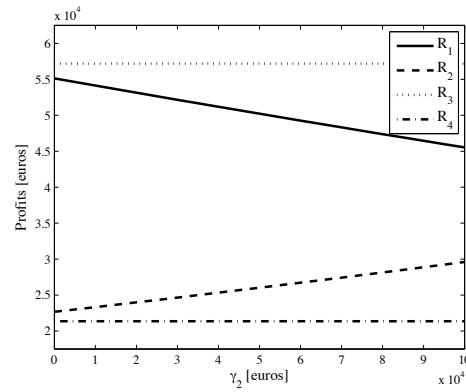
We have proposed a game theoretical framework to analyze the interactions between retailers and consumers in an electricity market.

Retailers seek to maximize their profit by selecting the optimal prices to offer to the consumers. This is achieved by anticipating the optimal response of the consumers, which maximize their utility

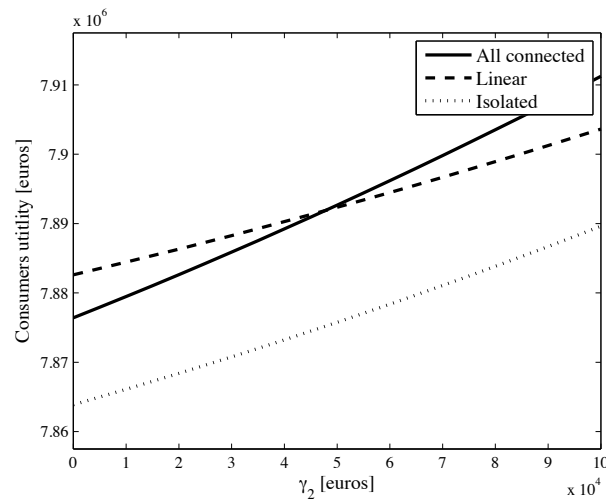


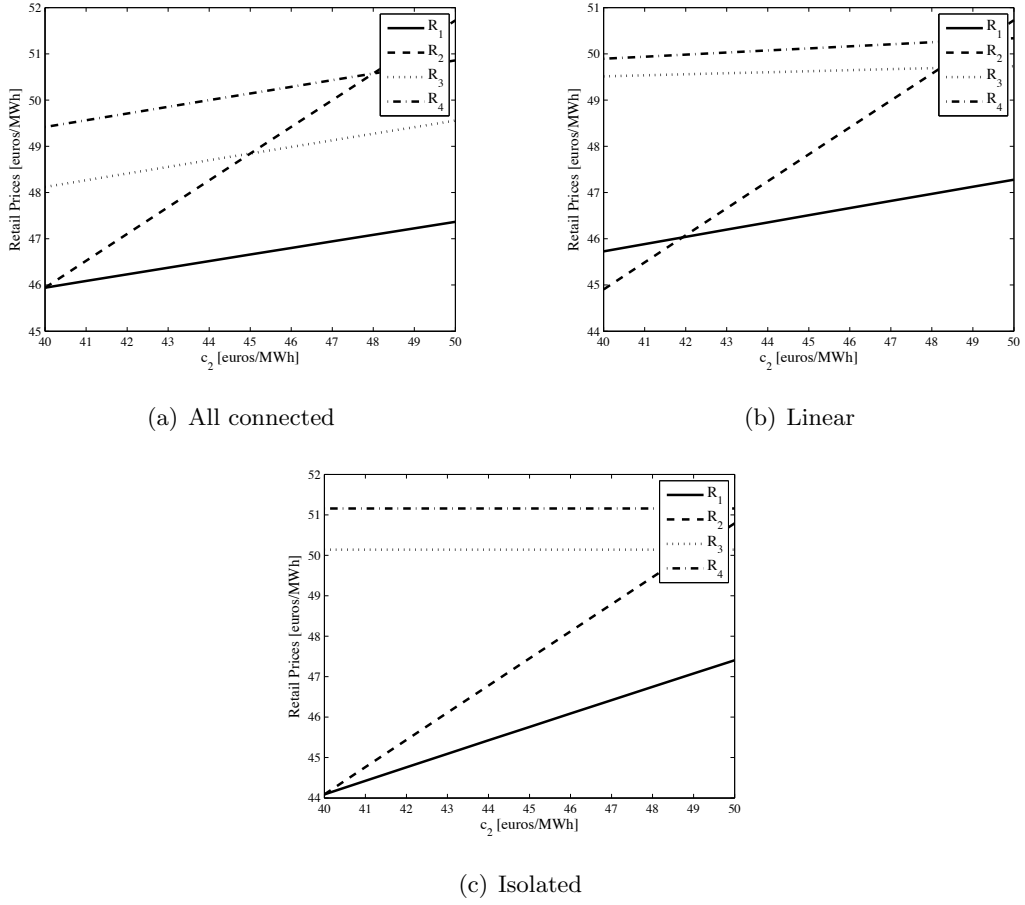
(a) All connected

(b) Linear



(c) Isolated

**Figure 9** Impact of  $\gamma_2$  on retailer's profits.**Figure 10** Impact of  $\gamma_2$  on consumers' utility.



**Figure 11** Impact of  $c_2$  on prices

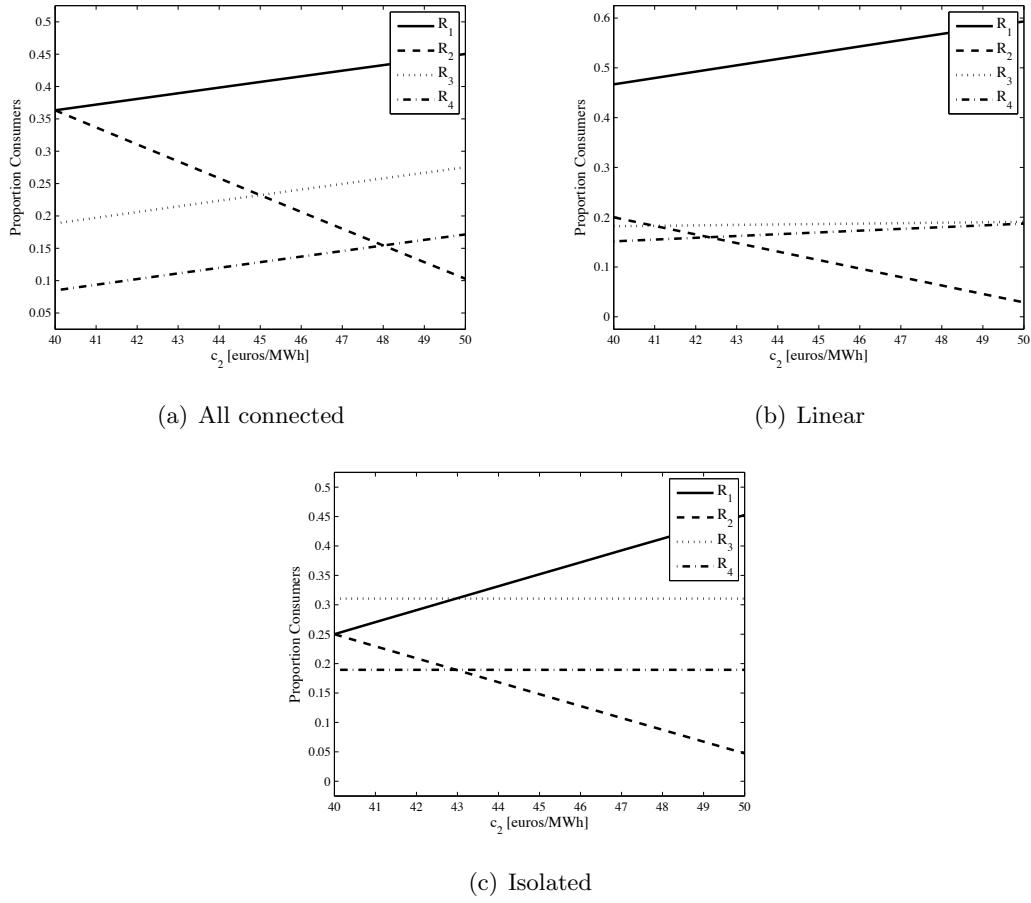
by deciding their optimal load quantity. Additionally, we account for the loyalty incentives that consumers may lose if they decide to switch from their original retailer. Uncertainty is incorporated to our model by assuming that consumer preferences for each retailer lay in a Hotelling line.

For a retailer duopoly, we analytically characterize the equilibrium, providing price sensitivities as well as existence and uniqueness conditions that hold for a wide class of consumer utility functions.

The duopoly is extended to a general case where several retailers located in a network compete simultaneously for groups of consumers.

We have performed an empirical analysis via numerical simulation to show how the different model parameters as well as the different network configurations impact the equilibrium market outcomes.

As highlights for the general duopoly case, and depending on the market conditions, the resulting price equilibrium can be classified into three categories: i) dominant position of a retailer; ii) leading position of a retailer; iii) complete competition; each of which with different market implications. The numerical simulations show how deviating from the complete competition case implies higher

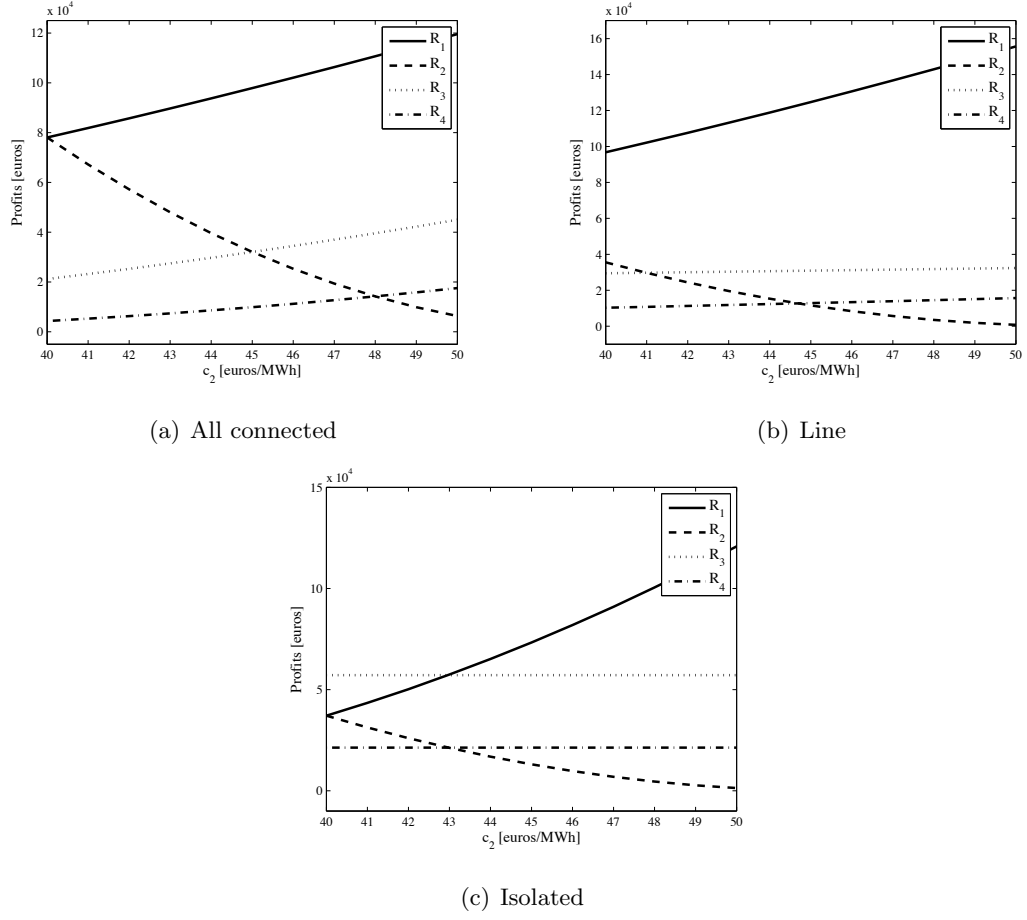


**Figure 12** Impact of  $c_2$  on market shares

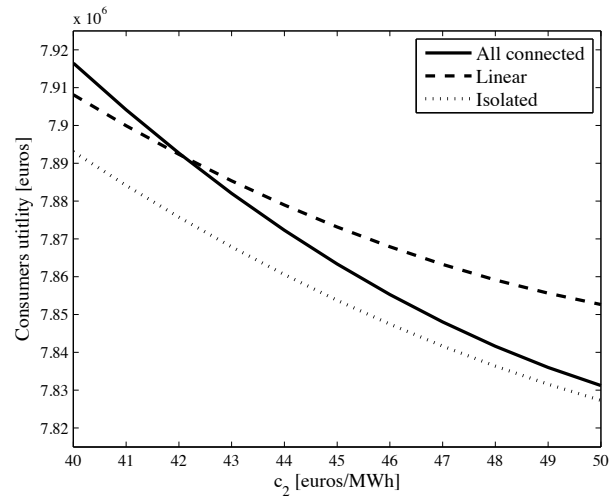
retail prices, higher retailer profits and lower consumer utilities. This deviation can be caused by an increase in a retailer's loyalty reward or by a decrease in a retailer costs. Moreover, results indicate that an increase in the initial market share of a leading retailer may be beneficial for the consumers as the market equilibrium may move to complete competition.

In the multiple-retailers case, a fully connected network, i.e., there exist groups of consumers that can select between any pair of retailers, entails that equilibrium prices are more homogeneous than in an incomplete network.

It should be noted that an increase of a retailer's loyalty reward increases the margin to raise its price while forcing its rivals to decrease theirs. Furthermore, if a retailer purchases its energy at higher costs, it is forced to set higher retail prices. However, in this case rival retailers benefit from this competitiveness decrease by also raising their retail prices. These effects are more relevant as the network is more connected. From a consumer's perspective, the worst network configuration is when retailers are isolated from each other. Under this setting, retailers can further exercise their market power over their corresponding consumers.



**Figure 13** Impact of  $c_2$  on retailer's profits



**Figure 14** Impact of  $c_2$  on consumers' utility

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## Proofs of Statements

### Proof for PROPOSITION 2:

From (8) and (10) we can check that, under condition (11), it holds that

$$\theta_{ij}^* > 1 \text{ and } \theta_{ji}^* \leq 0,$$

implying  $F_{ij}^* = 1$  and  $F_{ji}^* = 0$ , with satisfies Definition 2 of a dominant position of retailer  $R_i$ .

For retailer  $R_i$ , from (4) we get  $\pi_i^*(p_i, p_j) = 1$  and thus the optimal utility becomes

$$u_i^r(p_i, p_j) = (p_i - c_i)v^*(p_i) \quad (\text{EC.1})$$

and attains its maximum for a price  $p_i$  that satisfies

$$(p_i - c_i)v'(p_i) + v(p_i) = 0. \quad (\text{EC.2})$$

For retailer  $R_j$ , because  $\pi_j^*(p_j, p_i) = 0$ , the optimal utility vanishes and thus the profit regardless of the price  $p_j$ .  $\square$

### Proof for PROPOSITION 3:

From (8) and (10), we can check that under condition (13), we have

$$\theta_{ij}^* \geq 1 \text{ and } 0 < \theta_{ji}^* < 1,$$

implying  $F_{ij}^* = 1$  and  $F_{ji}^* = \theta_{ji}^*$ .

Hence, from (4) we get,  $\pi_i^*(p_i, p_j) = \pi_{ij} + (1 - \theta_{ji}^*)\pi_{ji}$  for retailer  $R_i$  and  $\pi_j^*(p_1, p_2) = \pi_{ji} \theta_{ji}^*$  for retailer  $R_j$ . The respective utilities become

$$u_i^r(p_i, p_j) = (p_i - c_i)v^*(p_i)(\pi_{ij} + (1 - \theta_{ji}^*)\pi_{ji}) \quad (\text{EC.3})$$

$$u_j^r(p_i, p_j) = (p_j - c_j)v^*(p_j)(\pi_{ji}\theta_{ji}^*). \quad (\text{EC.4})$$

By considering (8) and (10), the first order conditions associated to maximize (EC.3) and (EC.4), with respect to  $p_i$  and  $p_j$ , respectively, yields

$$(v(p_i) + (p_i - c_i)v'(p_i))(2k\pi_{ij} + \pi_{ji}(\varphi(p_i) - \varphi(p_j) + k - \gamma_j)) + \pi_{ji}(p_i - c_i)v(p_i)\varphi'(p_i) = 0$$

$$(v(p_j) + (p_j - c_j)v'(p_j))(2k\pi_{ji} - \pi_{ji}(\varphi(p_i) - \varphi(p_j) + k - \gamma_j)) + \pi_{ji}(p_j - c_j)v(p_j)\varphi'(p_j) = 0.$$

Using the definition (12) the above conditions can be rewritten as

$$\Gamma(p_i; c_i) = -\varphi(p_j) + \frac{1 + \pi_{ij}}{\pi_{ji}}k - \gamma_j$$

$$\Gamma(p_j; c_j) = -\varphi(p_i) + k + \gamma_j.$$

$\square$



### Proof for PROPOSITION 4:

Under condition (16) we have

$$0 < \theta_{ij}^* < 1 \text{ and } 0 < \theta_{ji}^* < 1,$$

implying  $F_{ij}^* = \theta_{ij}^*$  and  $F_{ji}^* = \theta_{ji}^*$ .

By using (4) we get  $\pi_i^*(p_i, p_j) = \theta_{ij}^* \pi_{ij} + (1 - \theta_{ji}^*) \pi_{ji}$ , and  $\pi_j^*(p_i, p_j) = \theta_{ji}^* \pi_{ji} + (1 - \theta_{ij}^*) \pi_{ij}$ . Thus, the profit function for each retailer becomes

$$u_i^r(p_i, p_j) = (p_i - c_i) v^*(p_i) (\theta_{ij}^* \pi_{ij} + (1 - \theta_{ji}^*) \pi_{ji}) \quad (\text{EC.5})$$

$$u_j^r(p_i, p_j) = (p_j - c_j) v^*(p_j) (\theta_{ji}^* \pi_{ji} + (1 - \theta_{ij}^*) \pi_{ij}). \quad (\text{EC.6})$$

By considering (8) and (10), the first order conditions associated to maximize (EC.5) and (EC.6), with respect to  $p_i$  and  $p_j$ , respectively, yield

$$(v(p_i) + (p_i - c_i) v'(p_i)) (\varphi(p_i) - \varphi(p_j) + k + \pi_{ij} \gamma_i - \pi_{ji} \gamma_j) + (p_i - c_i) v(p_i) \varphi'(p_i) = 0$$

$$(v(p_j) + (p_j - c_j) v'(p_j)) (\varphi(p_j) - \varphi(p_i) + k + \pi_{ji} \gamma_j - \pi_{ij} \gamma_i) + (p_j - c_j) v(p_j) \varphi'(p_j) = 0.$$

Using definition (12) the above optimal conditions can be rewritten as

$$\Gamma(p_i; c_i) = -\varphi(p_j) + k + \pi_{ij} \gamma_i - \pi_{ji} \gamma_j$$

$$\Gamma(p_j; c_j) = -\varphi(p_i) + k - \pi_{ij} \gamma_i + \pi_{ji} \gamma_j.$$

□

### Proof for THEOREM 1:

As  $\Gamma(c; c) = -\varphi(c)$ , under Condition C.4 and from Lemma 1 there always exists a value  $p_i \in [c_i; \tau(c_i))$  satisfying (23) and  $p_j \in [c_j; \tau(c_j))$  satisfying (24), given any value for  $p_j$  or  $p_i$  respectively. Furthermore, this value is unique.

For  $q \in [-\varphi(c); \infty)$  define

$$G(q; c) \in \Gamma^{-1}(q; c), \quad G(q; c) < \tau(c).$$

Lemma 1 implies  $G$  is a continuous and increasing function of  $q$  on  $[-\varphi(c); \infty)$ .

From this definition, a solution of

$$p_i = G(-\varphi(p_j) + \kappa_i; c_i) \quad (\text{EC.7})$$

$$p_j = G(-\varphi(p_i) + \kappa_j; c_j), \quad (\text{EC.8})$$

is also a solution of (23)–(24). Define

$$\omega_i \equiv G(-\varphi(\tau(c_j)) + \kappa_i; c_i), \quad \omega_j \equiv G(-\varphi(\tau(c_i)) + \kappa_j; c_j),$$

and note that for any  $p = (p_i, p_j)$  it will hold that  $p \in [c_i; \omega_i) \times [c_j; \omega_j)$ . Hence, by considering the property that  $-\varphi$  is an increasing function,  $\omega_k < \tau(c_k)$ , for  $k = \{i, j\}$ , and  $\Gamma(c; c) = -\varphi(c)$ , we have

$$\begin{aligned} -\varphi(c_i) &\leq -\varphi(c_j) + \kappa_i \leq -\varphi(p_j) + \kappa_i < -\varphi(\tau(c_j)) + \kappa_i \\ \Rightarrow c_i &\leq G(-\varphi(p_j) + \kappa_i; c_i) < \omega_i, \end{aligned}$$

with an equivalent bound holding for  $\omega_j$ .

As a consequence, system (EC.7)–(EC.8) defines a fixed-point iteration  $p = \Phi(p)$  where  $\Phi : [c_i; \omega_i) \times [c_j; \omega_j) \rightarrow [c_i; \omega_i) \times [c_j; \omega_j)$  and is continuous on that set. Brouwer's fixed-point theorem then implies the existence of a fixed point for  $\Phi$ , that will also be a solution for (23)–(24).  $\square$

## Proof for THEOREM 2:

Consider the fixed-point dynamics for the solution defined by (EC.7)–(EC.8), which we write in compact form as  $p = \Phi(p)$ , and define the fixed-point iteration

$$p = \Phi_j(\Phi_i(p)) \equiv \psi(p), \quad (\text{EC.9})$$

where  $\Phi = (\Phi_i \ \Phi_j)^T$  and  $\psi : [c_j; \tau(c_j)) \rightarrow [c_j; \tau(c_j))$ . This iteration is equivalent to (EC.7)–(EC.8), as any solution of (EC.9) provides a solution for (EC.7)–(EC.8) by letting  $p_j = p$  and  $p_i = \Phi_i(p)$ , and a solution for (EC.9) can be obtained from (EC.7)–(EC.8) by letting  $p = p_j$ .

$G$  is differentiable wrt  $p$  on  $[-\varphi(c_k); \infty)$ ,  $k = i, j$ , and we have that

$$G'(-\varphi(p) + \kappa_i; c_i) = \frac{v(p)}{\Gamma'(G(-\varphi(p) + \kappa_i; c_i); c_i)} \quad (\text{EC.10})$$

$$G'(-\varphi(\Phi_i(p)) + \kappa_j; c_j) = \frac{v(\Phi_i(p))}{\Gamma'(G(-\varphi(\Phi_i(p)) + \kappa_j; c_j); c_j)}. \quad (\text{EC.11})$$

Using the expression for  $\Gamma'$  in (22), we can write

$$\Gamma'(p; c) = v(p)(1 + \rho(p; c)), \quad (\text{EC.12})$$

for  $\rho$  defined in (25). We can rewrite (EC.10)–(EC.11) as

$$G'(-\varphi(p) + \kappa_i; c_i) = \frac{v(p)}{v(G(-\varphi(p) + \kappa_i; c_i))} \frac{1}{1 + \rho(G(-\varphi(p) + \kappa_i; c_i); c_i)} \quad (\text{EC.13})$$

$$\begin{aligned} G'(-\varphi(\Phi_i(p)) + \kappa_j; c_j) &= \frac{v(\Phi_i(p))}{v(G(-\varphi(\Phi_i(p)) + \kappa_j; c_j))} \\ &\quad \times \frac{1}{1 + \rho(G(-\varphi(\Phi_i(p)) + \kappa_j; c_j); c_j)}. \end{aligned} \quad (\text{EC.14})$$

From (EC.9) and using (EC.7)–(EC.8), replacing (EC.13)–(EC.14) and  $\Phi_i(p) = G(-\varphi(p) + \kappa_i; c_i)$  we have

$$\begin{aligned} \psi'(p) &= \frac{v(p)}{v(G(-\varphi(\Phi_i(p)) + \kappa_j; c_j))} \\ &\quad \times \frac{1}{1 + \rho(G(-\varphi(\Phi_i(p)) + \kappa_j; c_j); c_j)} \frac{1}{1 + \rho(G(-\varphi(p) + \kappa_i; c_i); c_i)}. \end{aligned}$$

From Condition C.3', for all  $p \in [\bar{c}_k; \tau(c_k))$  and  $k = i, j$  it holds that  $1 + \rho(p; c_k) > \sqrt{v(\bar{c})/v(\bar{\tau})}$ , and as we also have  $v(\bar{c}) \geq v(\bar{c}_j) \geq v(p) \geq v(\tau(c_j)) \geq v(\bar{\tau})$ , it follows that

$$\psi'(p) < \frac{v(\bar{c}_j)}{v(\tau(c_j))} \frac{v(\bar{\tau})}{v(\bar{c})} < 1,$$

implying  $1 - \psi'(p) > 0$ .

From this bound,  $p - \varphi(p)$  is strictly increasing on  $[\bar{c}_j; \tau(c_j))$ , there can only be one zero of the function in that interval and the fixed point for (EC.9) is unique on  $[\bar{c}_i; \tau(c_i)) \times [\bar{c}_j; \tau(c_j))$ .  $\square$